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Essential manifolds with extra structures

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ABSTRACT

We consider classes of algebraic manifolds $\mathcal{A}$, of symplectic manifolds $\mathcal{S}$, of symplectic manifolds with the hard Lefschetz property $\mathcal{HS}$ and the class of cohomologically symplectic manifolds $\mathcal{CS}$. For every class of manifolds $\mathcal{C}$ we denote by $\mathcal{EC}(\pi, n)$ a subclass of n-dimensional rationally essential manifolds with fundamental group π. In this paper we prove that for all the above classes with symplectically aspherical form the condition $\mathcal{EC}(\pi, 2n) \neq \emptyset$ implies that $\mathcal{EC}(\pi, 2n-2) \neq \emptyset$ for every $n > 2$. Also we prove that all the inclusions $\mathcal{EA} \subset \mathcal{EHS} \subset \mathcal{ES} \subset \mathcal{ECS}$ are proper.

1. Introduction

Let M be a closed, connected, orientable manifold of dimension n and let π be the fundamental group of M. Recall that a map $f : M \to K(\pi, 1)$ is called a *classifying map* for M if f induces an isomorphism $f_* : \pi_1(M, x_0) \to \pi_1(K(\pi, 1), f(x_0))$ for all $x_0 \in M$. It is well known that a classifying map exists and is unique up to homotopy. Gromov called a manifold M *inessential* if there exists a classifying map $f : M \to K(\pi, 1)$ to the $(n-1)$-skeleton of $K(\pi, 1)$. Otherwise M is called *essential* [12]. Gromov noticed that manifolds with positive scalar curvature tend to be inessential. He introduced several classes of essential manifolds (hyperspherical, hypereuclidean, enlargeable, [13]) for which he jointly with Lawson proved that manifolds of those classes cannot carry a metric with positive scalar curvature [16]. The following is found in reference [9, Lemma 2.4].

1.1. Proposition. *An orientable n-manifold M is* essential *if and only if the homomorphism $f_* : H_n(M) \to H_n(K(\pi, 1))$ induced by the classifying map f is nontrivial. Equivalently, if the image of the fundamental class $[M] \in H_n(M)$ under f_* is nontrivial in the nth integral homology group $H_n(K(\pi, 1))$ of the Eilenberg–MacLane space $K(\pi, 1)$.*

For example, the real projective space $\mathbb{RP}^{2n+1}$ is an essential manifold. Every aspherical manifold (for example, the torus $\mathbb{T}^n$, a compact orientable surface M_g of genus g) is essential. There are no simply connected essential manifolds of positive dimension.

1.2. Definition. Let M be a closed, connected, orientable manifold of dimension n and let π be the fundamental group of M. We say that manifold M is *rationally essential* if a classifying map $f : M \to K(\pi, 1)$ induces nontrivial homomorphism $f_* : H_n(M; \mathbb{Q}) \to H_n(K(\pi, 1); \mathbb{Q})$.

Clearly, every rationally essential manifold is essential but not vise versa: $\mathbb{RP}^{2n+1}$ is not rationally essential.

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Clearly, if $H_n(K(\pi,1))=0$ then there are no essential (and hence rationally essential) n-manifolds with the fundamental group π. The converse also holds: if π is a finitely presented group and $H_n(\pi;\mathbb{Q})\neq 0$ then there exists a rationally essential n-manifold with the fundamental group π, see Theorem 2.1 below.

Brunnbauer and Hanke gave a characterization of Gromov type classes of rationally essential manifolds with given fundamental group in terms of group homology [3]. In this paper we consider similar problem for some symplectic type classes.

Given a class of manifolds $\mathcal{C}$ we denote by $\mathcal{EC}$ the subclass that consists of rationally essential manifolds. Here we consider the following classes:

$$\mathcal{A}\subset\mathcal{HS}\subset\mathcal{S}\subset\mathcal{CS}$$

where $\mathcal{A}$ is the class of algebraic manifolds, $\mathcal{S}$ is the class of symplectic manifolds, $\mathcal{HS}$ is the class of symplectic manifolds with the hard Lefschetz property, and $\mathcal{CS}$ is the class of cohomologically symplectic manifolds (see Sections 2 and 3 below). It is known that all the above inclusions of classes are proper [4,26,10,7]. We will show that the inclusions of the essential counterparts are also proper.

For every class of manifolds $\mathcal{C}$ we denote by $\mathcal{C}(\pi,n)$ a subclass of n-dimensional manifolds with fundamental group π. This paper is an attempt to study the following question.

Main problem. For which values π and n, is $\mathcal{EC}(\pi,n)$ non-empty?

In particular, in the paper we address the following conjecture proposed by Dranishnikov and Rudyak:

Conjecture. *For the first three above classes for $n>2$ the condition $\mathcal{EC}(\pi,2n)\neq\emptyset$ implies that $\mathcal{EC}(\pi,2n-2)\neq\emptyset$.*

We prove for all the above classes a weaker version of the conjecture that deals with symplectically aspherical manifolds, see Section 3 for the definition.

Note that every complex projective algebraic manifold V is symplectic: the corresponding symplectic form is given by the Kähler form [15, p. 109]. In particular, we are able to speak about symplectically aspherical algebraic manifolds.

2. Preliminaries

The following fact is known (see for example [3,8]). Since there is no detailed proof of it in print, we give a complete proof here.

2.1. Theorem. *For every finitely presented group π and every integer n if $H_n(\pi;\mathbb{Q})\neq 0$ then for every nontrivial element $\alpha\in H_n(\pi;\mathbb{Q})$ there exists a closed, connected, orientable n-manifold M, an integer $k\neq 0$ and a map $f:M\to K(\pi,1)$ such that $f_*([M])=k\alpha$ and $f_*:\pi_1(M)\to\pi_1(K(\pi,1))$ is a group isomorphism.*

Proof. Let π be a finitely presented group and let n be an integer such that $H_n(\pi;\mathbb{Q})\neq 0$. Take any nontrivial element α in $H_n(\pi;\mathbb{Q})$. Because of a theorem of Thom, there exist a closed n-manifold N, an integer $k\neq 0$ and a map $g:N\to K(\pi,1)$ such that $g_*([N])=k\alpha$, see e.g. [22, Theorem IV.7.36]. Suppose that $g_*:\pi_1(N)\to\pi_1(K(\pi,1))$ is not surjective. Let $\alpha:[0,1]\to K(\pi,1)$ be a loop such that $[\alpha]\in\pi_1(K(\pi,1))\backslash im(g_*)$ and $\alpha(0)=\alpha(1)=y_0$. Without loss of generality we can assume that $y_0\in\operatorname{Im}(g)$ since the fundamental groups of $K(\pi,1)$ based at different points are isomorphic because $K(\pi,1)$ is path connected. Take $x_0\in N$ such that $g(x_0)=y_0$. Consider chart (U,φ) on N such that $\varphi(U)=\mathbb{R}^n$ and $\varphi(x_0)=0$. Now define function $h:\mathbb{R}^n\to\mathbb{R}^n$ in generalized spherical coordinates as follows

$$h(r,\theta_1,\dots,\theta_{n-1})=\begin{cases}0, & \text{if } 0\leqslant r\leqslant 1,\\ (r-1,\theta_1,\dots,\theta_{n-1}), & \text{if } r>1.\end{cases}$$

To perform a surgery on a manifold N we shall define a new function $\widetilde{g}$ by: $\widetilde{g}(x)=g(x)$ if $x\notin U$, $\widetilde{g}(x)=g(\varphi^{-1}h\varphi(x))$ if $x\in U$. Then $\widetilde{g}$ is homotopic to g because h is homotopic to the identity map on $\mathbb{R}^n$. Let D be the preimage under φ of the unit ball in $\mathbb{R}^n$ centered at 0. Now we perform a surgery on the manifold N. There exists an embedding $i:S^0\times D^n\to N$ such that $i(S^0\times D^n)\subseteq D$ and $x_0\notin i(S^0\times D^n)$. Form a new manifold from the union of $N\times I$ and $D^1\times D^n$ by attaching $S^0\times D^n$ to its image under $i\times 1$. We can extend map $\widetilde{g}\times 1$ by defining $\widetilde{g}$ on $D^1\times D^n$ as follows

$$\widetilde{g}(t,x)=\alpha(t)\quad\text{for all } (t,x)\in D^1\times D^{n-1}.$$

Connect point x_0 with points $(0,c),(1,c)$ in $D^1\times D^{n-1}$ for some $c\in D^n$ by paths $\gamma_1(t),\gamma_2(t)$ respectively. Let $\beta(t)=(t,c)\in D^1\times D^n$ for all $t\in[0,1]$. Then $(\widetilde{g}\times 1)_*(\gamma_1\beta\gamma_2^{-1})=\alpha$. So we can construct a manifold $\widetilde{N}$ and a map $\widetilde{g}:\widetilde{N}\to K(\pi,1)$ such that $\widetilde{g_*}([\widetilde{N}])=k\alpha$ and $\widetilde{g}_*$ induces an epimorphism on fundamental groups. Now we want to perform surgeries that annihilate the elements that generate the kernel of $\widetilde{g_*}$. Note that since $\widetilde{N}$ is orientable then every loop γ in $\widetilde{N}$ can be homotoped to a loop $\widetilde{\gamma}$ that has trivial normal bundle in $\widetilde{N}$. Clearly, if a loop $\widetilde{\gamma}$ is trivial then the loop $\alpha\widetilde{\gamma}\alpha^{-1}$ is also trivial for every path $\alpha:[0,1]\to\widetilde{N}$ such that $\alpha(1)=\widetilde{\gamma}(0)$. Since $\operatorname{Ker}(\widetilde{g_*})$ is normally finitely generated [28] then we can perform surgery on $\widetilde{N}$ finitely many times to construct a manifold M and a map $f : M \to K(\pi, 1)$ that induces isomorphism $f_* : \pi_1(M) \to \pi_1(K(\pi, 1))$ and such that $f_*([M]) = k\alpha$. $\square$

Note that every oriented manifold of dimension $\leqslant 2$ is essential, an oriented 3-manifold M is essential iff the group $\pi_1(M)$ is not free, [14,24].

2.2. Definition. We define a cohomology class $v \in H^m(X; G)$ to be *aspherical* if $v = f^*(a)$ for a classifying map $f : X \to K(\pi_1(X), 1)$ and some $a \in H^m(K(\pi_1(X), 1); G)$.

Note that if a class v is aspherical and $v^k \neq 0$ then v^k is aspherical.

2.3. Proposition. *Let M be a closed, orientable manifold of dimension km, and let $u \in H^m(M; \mathbb{Q})$ be an aspherical class. If $u^k \neq 0$, then M is rationally essential.*

2.4. Definition. A *symplectic structure* on a smooth manifold M is a non-degenerate skew-symmetric closed 2-form $\omega \in \Omega^2(M)$. A *symplectic manifold* is a pair (M, ω) where M is a smooth manifold and ω is a symplectic structure on M.

The non-degeneracy condition means that for all $p \in M$ we have the property that if $\omega(v, w) = 0$ for all $w \in T_pM$ then $v = 0$. The skew-symmetry condition means that for all $p \in M$ we have $\omega(v, w) = -\omega(w, v)$ for all $v, w \in T_pM$. The closed condition means that the exterior derivative $d\omega$ of ω is identically zero. Since each odd-dimensional skew-symmetric matrix is singular, we see that M has even dimension. Every symplectic $2n$-dimensional manifold (M, ω) is orientable since the n-fold wedge product $\omega \wedge \cdots \wedge \omega$ never vanishes.

2.5. Definition. A symplectic manifold (M, ω) is *symplectically aspherical* if

$$\int_{S^2} f^*\omega = 0$$

for every smooth map $f : S^2 \to M$.

Clearly, if $\pi_2(M) = 0$ then a symplectic manifold (M, ω) is symplectically aspherical. However there are symplectically aspherical manifolds with nontrivial π_2, [11,18].

2.6. Remark. The cohomology class $[\omega]$ in a symplectically aspherical manifold (M, ω) is aspherical. It follows from classical results of Hopf, [17] (see also [5, Theorem 8.17], [1, Theorem 5.2]).

In view of this remark and Proposition 2.3, we have the following corollary

2.7. Corollary. *Every closed symplectically aspherical manifold is rationally essential.*

To proceed, we need the following theorems, see e.g. [20, p. 41].

2.8. Theorem *(Lefschetz Hyperplane Theorem). Let V be a complex projective algebraic variety of complex dimension k which lies in the complex projective space $\mathbb{C}P^n$, and let P be a hyperplane in $\mathbb{C}P^n$ which contains the singular points (if any) of V. Then the relative homotopy groups $\pi_r(V, V \cap P)$ are equal to zero for all $r < k$.*

Note that $V \cap P$ is a manifold (i.e. non-singular variety) if V is.

2.9. Theorem *(Donaldson [6]). Let $L \to V$ be a complex line bundle over a compact symplectic manifold (V, ω) with compatible almost-complex structure, and with the first Chern class $c_1(L) = [\frac{\omega}{2\pi}]$. Then there is a constant C such that for all large k there is a section s of $L^{\otimes k}$ with*

$$|\bar{\partial} s| < \frac{C}{\sqrt{k}} |\partial s| \tag{2.1}$$

on the zero set of s.

2.10. Theorem *(Donaldson [6]). Let W_k be the zero-set of a section s of $L^{\otimes k} \to V$ satisfying the conditions of Theorem 2.9. When k is sufficiently large the inclusion $i : W_k \to V$ induces an isomorphism on homotopy groups π_p for $p \leqslant n - 2$ and a surjection on π_{n-1}.*

In view of Theorem 2.9 and Theorem 2.10 we obtain the following corollary

2.11. Corollary. *Let (M,ω) be a closed symplectic manifold of dimension $2n$ such that the cohomology class $[\omega]$ is integral. Then there exists a symplectic submanifold V of M of codimension 2 such that inclusion $i: V\to M$ induces an isomorphism on homotopy groups π_p for $p\leqslant n-2$ and a surjection on π_{n-1}. Furthermore, the homology class $[V]$ in M is the Poincaré dual to a class $r[\omega]$ for some integer r.*

Proof. The proof follows from Theorem 2.9 and Theorem 2.10 with ω normalized such that $c_1(L)=[\omega]$. Let V be the zero-set of a section s of $L^{\otimes k}\to M$ as in Theorem 2.9. Then inequality (2.1) guarantees the existence of symplectic structure on V. So V is a symplectic submanifold of M of codimension 2. The homology class of V is Poincaré dual to the first Chern class of $L^{\otimes k}$ up to a multiplicative constant r. Finally, according to Theorem 2.10 the inclusion $i: V\to M$ induces an isomorphism on homotopy groups π_p for $p\leqslant n-2$ and a surjection on π_{n-1}. □

3. Classes of essential manifolds

3.1. Theorem. *Assume that M is a complex projective algebraic manifold of (real) dimension $2k$ which lies in the complex projective space $\mathbb{C}P^N$. Suppose also that M is symplectically aspherical. Then for every integer m with $2\leqslant m\leqslant k$ there exists a rationally essential algebraic manifold V of dimension $2m$ with fundamental group isomorphic to $\pi_1(M)$.*

Proof. The case $m=k$ is Corollary 2.7. By induction, it suffices to prove the theorem for $m=k-1$. Indeed, assume that $\dim M=2k>4$ and let $V=M\cap\mathbb{C}P^{N-1}$. If we prove that V is a rationally essential complex algebraic manifold with $\dim V=2k-2>4$ and the fundamental group $\pi=\pi_1(M)$, we apply the previous argument for V instead of M. Because of Theorem 2.8, $\pi_r(M,V)=0$ for $r<k-1$. From the exactness of the homotopy sequence

$$\pi_2(M,V)\to\pi_1(V)\to\pi_1(M)\to\pi_1(M,V)$$

it follows that

$$\pi_1(M)\simeq\pi_1(V)\simeq\pi\quad\text{since } \pi_2(M,V)\simeq\pi_1(M,V)\simeq 0.$$

Hence V is a complex algebraic manifold with fundamental group isomorphic to π, and $\dim V=\dim M-2$.

It remains to prove that V is rationally essential. But this follows from Corollary 2.7 because the induced Kähler form on V is aspherical. □

3.2. Theorem. *Let (M,ω) be a closed symplectically aspherical manifold of dimension $2n>2$ with fundamental group π. Then for every k such that $2\leqslant k\leqslant n$ there exists a symplectically aspherical manifold V of dimension $2k$ with fundamental group isomorphic to π.*

Proof. We prove the theorem by induction. Similarly to the proof of Theorem 3.1, it suffices to prove the case $k=n-1$. Without loss of generality, we can assume that the cohomology class $[\omega]$ is integral (see [18, Prop. 1.5]). Let M be a manifold as in Corollary 2.11. Then, for $n>2$, the inclusion $i: V\to M$ induces an isomorphism on the fundamental groups $\pi_1(V)\to\pi_1(M)$. Now, V is a symplectic manifold with symplectic structure $i^*\omega$ induced from M. It is clear that

$$\int_{S^2} g^*i^*\omega=0$$

for every map $g: S^2\to V$. Thus $(V,i^*\omega)$ is a symplectically aspherical manifold of dimension $2n-2$ with $\pi_1(V)=\pi$. □

3.3. Definition. A symplectic manifold (M^{2n},ω) has the *hard Lefschetz property* (HLP) if the map

$$L^k_{[\omega]}: H^{n-k}_{DR}\big(M^{2n}\big)\to H^{n+k}_{DR}\big(M^{2n}\big),\qquad L^k_{[\omega]}\big([x]\big)=\big[\omega^k\wedge x\big]$$

is an isomorphism for all $k=0,\dots,n$.

For example, the Hard Lefschetz Theorem says that every Kähler manifold has HLP, see [15, p. 122].

3.4. Theorem. *Let (M,ω) be a symplectically aspherical manifold of dimension $2n>2$ with fundamental group π and having HLP. Then for every m such that $2\leqslant m\leqslant n$ there exists a symplectically aspherical manifold (V,η) of dimension $2m$ with fundamental group isomorphic to π and having HLP.*

Proof. We follow the proof of Theorem 3.2 and must prove that the manifold V as in Theorem 3.2 has HLP.

First, we need to show that $L^k_{[\omega^*]}: H^{n-1-k}(V)\to H^{n-1+k}(V)$ is an isomorphism for all $k=0,\dots,n-1$ where ω^* is the pullback of ω under inclusion $i: V\to M$. We need to consider separately the case when $k=0$. So fix any k such that

$0 < k \leqslant n-1$. Since $H^{n-1-k}(V)$ and $H^{n-1+k}(V)$ have the same dimension, it suffices to show that $L^k_{[\omega*]}$ is a monomorphism. Consider the following commutative diagram

$$\begin{tikzcd} H^{n-1-k}(M) \arrow[r, "L^k_{[\omega]}"] \arrow[d, "i_1^*"'] & H^{n-1+k}(M) \arrow[r, "\smile\omega"] \arrow[d, "i_2^*"] & H^{n+1+k}(M) \\ H^{n-1-k}(V) \arrow[r, "L^k_{[\omega^*]}"] & H^{n-1+k}(V) & \end{tikzcd}$$

where $L^k_{[\omega]}$ is a monomorphism because $L^{k+1}_{[\omega]}$ is an isomorphism. It follows from Corollary 2.11, and Whitehead theorem (see [25, p. 399]) that i_1^* is an isomorphism. Hence it suffices to show that i_2^* is a monomorphism on the $\operatorname{Im}(L^k_{[\omega]})$. Assume that $\alpha \in H^{n-1-k}(M)$ is nontrivial and $i_2^*(\alpha \smile \omega^k) = 0$. Then

$$\begin{aligned} 0 &\neq r\bigl([M] \frown \bigl(\alpha \smile \omega^{k+1}\bigr)\bigr) = r\bigl([M] \frown \bigl(\alpha \smile \omega^k\bigr)\bigr) \frown \omega \\ &= r\bigl([M] \frown \omega\bigr) \frown \bigl(\alpha \smile \omega^k\bigr) = i_*\bigl([V]\bigr) \frown \bigl(\alpha \smile \omega^k\bigr) \\ &= i_*\bigl([V] \frown i_2^*\bigl(\alpha \smile \omega^k\bigr)\bigr) = 0. \end{aligned}$$

This is a contradiction. So $L^k_{[\omega^*]}$ is an isomorphism for all $k = 1, \dots, n-1$.

If $k = 0$ then it is obvious that $L^0_{[\omega^*]} : H^{n-1}(V) \to H^{n-1}(V)$ is an isomorphism. Thus V is a symplectically aspherical manifold of dimension $2n-2$ with fundamental group π having the HLP.

Now we can apply the above procedure to V, and the result follows by induction. □

3.5. Definition *(Lupton–Oprea [19]).* A manifold M of dimension $2n$ is *cohomologically symplectic* (or, briefly, c-symplectic) if there exists a closed differential 2-form ω on M such that $[\omega]^n \neq 0$.

Clearly, not all c-symplectic manifolds are symplectic. For example, $\mathbb{C}P^2 \# \mathbb{C}P^2$ is c-symplectic but is not symplectic [10].

3.6. Theorem. *Let (M, ω) be a c-symplectic manifold of dimension $2n > 2$ with fundamental group π and with aspherical c-symplectic form. Then for every m such that $2 \leqslant m \leqslant n$ there exists a c-symplectic manifold (V, η) of dimension $2m$ with fundamental group isomorphic to π and with aspherical c-symplectic form.*

Proof. Let $f : M \to K(\pi, 1)$ be a classifying map for M. Then $\omega = f^*a$ for some $a \in H^2(K(\pi, 1))$. There exists a $(2n-2)$-dimensional submanifold N of M such that $[N] = r\eta$ for some $r \in \mathbb{Z}$, where $\eta = PD([\omega]) = [M] \frown \omega$. Let $i : N \to M$ be the inclusion of N into M. We want to show that $(i^*\omega)^{n-1} \neq 0$. Suppose that $(i^*\omega)^{n-1} = 0$. Then

$$\begin{aligned} 0 &\neq r\bigl([M] \frown \omega^n\bigr) = r\bigl([M] \frown \omega \frown \omega^{n-1}\bigr) \\ &= i_*\bigl([N]\bigr) \frown \omega^{n-1} = i_*\bigl([N] \frown \bigl(i^*\omega\bigr)^{n-1}\bigr) = 0. \end{aligned}$$

This is a contradiction. Hence $(i^*\omega)^{n-1} \neq 0$. By using surgery we can construct a manifold N' and a map $i' : N' \to M$ that induces an isomorphism on the fundamental groups. Moreover, there exist a manifold W with $\partial W = N \sqcup N'$ and a map $g : W \to M$ that extends i and i'. In other words, the singular manifolds $i : N \to M$ and $i' : N' \to M$ are bordant:

$$\begin{tikzcd} N \arrow[r, "j"] \arrow[dr, "i"'] & W \arrow[d, "g"] & N' \arrow[l, "j'"'] \arrow[dl, "i'"] \\ & M & \end{tikzcd}$$

where j and j' are the inclusions. Thus $i'_*([N']) = i_*([N])$. Now

$$\begin{aligned} \bigl\langle \bigl(i'^*\omega\bigr)^{n-1}, \bigl[N'\bigr] \bigr\rangle &= \bigl\langle \omega^{n-1}, i'_*\bigl(\bigl[N'\bigr]\bigr) \bigr\rangle \\ &= \bigl\langle \omega^{n-1}, i_*\bigl([N]\bigr) \bigr\rangle = \bigl\langle \bigl(i^*\omega\bigr)^{n-1}, [N] \bigr\rangle \neq 0, \end{aligned}$$

so $(i'^*\omega)^{n-1} \neq 0$. Thus $(N', i'^*\omega)$ is a c-symplectic manifold of dimension $2n-2$ with fundamental group isomorphic to π. Clearly, $i'^*\omega$ is an aspherical form because $i'^*\omega = (f \circ i')^*a$. The result follows by induction. □

3.7. Proposition. *There is an example of a rationally essential 4-dimensional c-symplectic manifold M which is not symplectic.*

Proof. Let Σ be an aspherical 4-dimensional homology sphere (see [23]). We consider the connected sum $M = \mathbb{C}P^2\#\mathbb{C}P^2\#\Sigma$ and show that it does not admit an almost complex structure. According to the result of Ehresmann and Wu, a compact 4-manifold M has an almost complex structure with first Chern class $c_1 \in H^2(M,\mathbb{Z})$ if and only if c_1 reduces modulo 2 to the second Stiefel–Whitney class w_2 and

$$c_1^2\bigl([M]\bigr) = 3\tau + 2\chi,$$

where χ is the Euler characteristic of M and τ is its signature ([21, p. 119]). A routine computation shows that $\chi = 4$, $\tau = 2$ and $c_1^2([M])$ is the sum of squares of two integers. But 14 cannot be represented in such form. Hence M does not admit an almost complex structure and therefore is not a symplectic manifold because every symplectic manifold admits a compatible almost complex structure. Furthermore, $\Sigma = K(\pi_1(\Sigma),1)$, and the collapsing map $f : M \to \Sigma$ has degree 1. Thus M is a rationally essential manifold since the homomorphism induced by f on the 4th homology groups $f_* : H_4(M;\mathbb{Q}) \to H_4(\Sigma;\mathbb{Q})$ is nontrivial.

Since Σ is a homology sphere, the collapsing map $i : M \to \mathbb{C}P^2\#\mathbb{C}P^2$ induces the isomorphism

$$i^* : H^2\bigl(\mathbb{C}P^2;\mathbb{R}\bigr) \oplus H^2\bigl(\mathbb{C}P^2;\mathbb{R}\bigr) \to H^2(M;\mathbb{R}).$$

Let $\{[\omega_1],[\omega_2]\}$ be a basis of $H^2(\mathbb{C}P^2;\mathbb{R}) \oplus H^2(\mathbb{C}P^2;\mathbb{R})$. Then $i^*([\omega_1]+[\omega_2])^2 \neq 0$ in $H^4(M;\mathbb{R})$. Hence M is a c-symplectic manifold. □

3.8. Remark. Note that the Dranishnikov–Rudyak conjecture is not true for c-symplectic manifolds. Consider a rationally essential c-symplectic manifold $M = \mathbb{C}P^4\#\mathbb{C}P^4\#(\Sigma\times\Sigma)$ with fundamental group $\pi_1(M) \simeq \pi_1(\Sigma)\times\pi_1(\Sigma)$. Since $\Sigma\times\Sigma$ is the Eilenberg–MacLane space $K(\pi_1(M),1)$ and $H_6(\Sigma\times\Sigma;\mathbb{Q})$ is trivial then there does not exist a rationally essential 6-manifold with fundamental group isomorphic to $\pi_1(M)$.

3.9. Theorem. *All the inclusions of classes*

$$\mathcal{EA} \subset \mathcal{EHS} \subset \mathcal{ES} \subset \mathcal{ECS}$$

are proper.

Proof. First we prove that the inclusion $\mathcal{EA} \subset \mathcal{EHS}$ is proper. Let $\mathbb{H}$ be the Heisenberg manifold. Then the blow-up M of $\mathbb{H}\times\mathbb{H}$ along a torus is a symplectic manifold that satisfies the hard Lefschetz property and has nontrivial triple Massey product [4]. Since $\mathbb{H}$ is an aspherical manifold then $\mathbb{H}\times\mathbb{H}$ is the Eilenberg–MacLane space. So M is a rationally essential manifold because there exists a degree 1 (classifying) map $f : M \to \mathbb{H}\times\mathbb{H}$. Note that M is not algebraic since it has non-trivial Massey product, while all Kähler (and therefore algebraic) manifolds are formal spaces, [7], and hence all their Massey products are trivial.

Now we prove that the inclusion $\mathcal{EHS} \subset \mathcal{ES}$ is proper. Consider the Kodaira–Thurston manifold KT obtained by taking the product of the Heisenberg manifold $\mathbb{H}$ and the circle S^1. It is well known that KT is a symplectic manifold. The Kodaira–Thurston manifold is rationally essential because it is a nilmanifold and it can not have the hard Lefschetz property because a symplectic nilmanifold of Lefschetz type is diffeomorphic to a torus [2].

We have already shown that the inclusion $\mathcal{ES} \subset \mathcal{ECS}$ is proper, see Proposition 3.7 above. □

The Dranishnikov–Rudyak conjecture cannot be reduced to the aspherical case in view of the following

3.10. Proposition. *The blow up of a 4-torus at a single point $M = T^4\#\overline{\mathbb{C}P^2}$ is an algebraic manifold which does not admit an aspherical symplectic form.*

Proof. Let ω be a symplectic form on M. Then $\int_M \omega^2 \neq 0$. We can obtain a form ω' on $\overline{\mathbb{C}P^2}$ that extends the restriction of ω on $\overline{\mathbb{C}P^2}\setminus D$ such that $\int_{\overline{\mathbb{C}P^2}} \omega'^2 \neq 0$ where D is a small enough disk. Then there exists a map $f : S^2 \to \overline{\mathbb{C}P^2}\setminus D$ with $\int_{S^2} f^*\omega' \neq 0$ because if we assume that $\int_{S^2} f^*\omega' = 0$ for all maps $f : S^2 \to \overline{\mathbb{C}P^2}\setminus D$ then $[\omega'] = 0$ in $H^2(\overline{\mathbb{C}P^2};\mathbb{R})$. Therefore $[\omega']^2 = 0$ and $\int_{\overline{\mathbb{C}P^2}} \omega'^2 = 0$ which contradicts to the choice of ω'. Consider $f : S^2 \to \overline{\mathbb{C}P^2}\setminus D$ such that $\int_{S^2} f^*\omega' \neq 0$. Since ω and ω' coincide on $\overline{\mathbb{C}P^2}\setminus D$ then $\int_{S^2} f^*\omega \neq 0$. Thus ω is not an aspherical symplectic form. □

It is natural to consider the class of Kähler manifolds $\mathcal{K}$ and ask whether the inclusions $\mathcal{EA} \subset \mathcal{EK} \subset \mathcal{ES}$ are proper. It is known that inclusions $\mathcal{A} \subset \mathcal{K} \subset \mathcal{S}$ are proper [27,4] and manifold M in Theorem 3.9 shows that inclusion $\mathcal{EK} \subset \mathcal{ES}$ is also proper. Note that M is not Kähler because it is not formal.

1. Question. Does there exist an essential Kähler manifold that is not algebraic?

2. Question. In view of the theorems proved above we may ask whether the Dranishnikov–Rudyak conjecture holds true for the class of Kähler manifolds with aspherical Kähler form.

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